H2 Mathematics Sequences: Arithmetic & Geometric Progression Summary

AP	GP
Example: a) $1,\ 1+\sqrt{2},\ 1+2\sqrt{2},\ 1+3\sqrt{2},\ \ldots$ b) $6,\ 2,\ -2,\ -6,\ \ldots,\ -30$	*Example:* a) $-\frac{1}{3},\frac{1}{2},-\frac{3}{4},\frac{9}{8},\ldots$ b) $1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \ldots$
Each term in the above number pattern can be written as: $u_n = a+(n-1)d$ **where:** u_n is the n^{th} term of an AP; a is the value of the first term; d is the common difference; n is the term number.	**Each term in the above number pattern can be written as:** $u_n = ar^{n-1}$ **where:** u_n is the n^{th} term of an GP; a is the value of the first term; r is the common ratio; n is the term number.
The sum of the **first *n*** terms of an AP is: $S_n = \frac{n}{2}\left[2a+(n-1)d\right]$ or $S_n = \frac{n}{2}(a+l)$ where a = first term d = common difference l = last term n = number of terms	The sum of the **first *n*** terms of an GP is: $S_n = \frac{a(1-r^n)}{1-r},\ r\neq 1$ **Sum to infinity** of an GP is: $S_\infty = u_1+u_2+u_3+\ldots = \frac{a}{1-r}$ $\left.\right\} S_n = (1-r^n)S_\infty$ *Provided that the GP is <u>convergent</u>, and <u>very importantly</u>, $-1<r<1$. where a = first term r = common ratio n = number of terms

Example:

An AP is such that its first term has a value of 2 and a common difference of 2. Given that the last term is 72, find the sum of all the terms in this series.

Solution:

$u_l = 2+(l-1)(2) = 72 \Rightarrow l = 36$

i.e. this series has a total of 36 terms.

Therefore,

$S_{36} = \dfrac{36}{2}\left[2(2)+(36-1)(2)\right] = 1332$ or $S_{36} = \dfrac{36}{2}(2+72) = 1332$

Example:

The sum to infinity of a geometric series is 3. When the terms of the geometric series are squared, a new geometric series is obtained where sum to infinity is $\dfrac{9}{2}$.

Find the first term and the common ratio of the first series.

Solution:

We have GP: $a, ar, ar^2, ar^3, \ldots$ with 1st term a and common ratio r.

Upon squaring: $a^2, a^2r^2, a^2r^4, a^2r^6, \ldots$ which is also a GP with 1st term a^2 and common ratio r^2

Given $S_\infty = \dfrac{a}{1-r} = 3$ ______(1) and $\dfrac{a^2}{1-r^2} = \dfrac{9}{2}$ ___________(2)

(2) $\div$ (1) gives $\dfrac{a}{1+r} = \dfrac{3}{2}$ ________(3)

(1) $\div$ (3) gives $\dfrac{1+r}{1-r} = 2 \Rightarrow 1+r = 2-2r$

i.e $3r = 1 \Rightarrow r = \dfrac{1}{3}$.

Hence, $a = 3\left(1-\dfrac{1}{3}\right) = 2$

Proving AP/GP

AP	GP
To prove that a number pattern is AP, show that: $u_n - u_{n-1} =$ common difference $=$ constant where $u_n = n^{th}$ term	To prove that a number pattern is GP, show that: $\dfrac{u_n}{u_{n-1}} =$ common ratio $=$ constant where $u_n = n^{th}$ term

Example

The n^{th} term of a sequence is $T_n = 7(2)^{2n-1}, n \in \mathbb{Z}^+$. Show that the sequence is a geometric progression.

Solution:

$T_n = 7(2)^{2n-1}$

$T_{n-1} = 7(2)^{2(n-1)-1} = 7(2)^{2n-3}$

$\Rightarrow \dfrac{T_n}{T_{n-1}} = \dfrac{7(2)^{2n-1}}{7(2)^{2n-3}} = 4$ which is a constant.

Hence, the sequence is a G.P.

Sometimes, we are given an expression of S_n in a question.

To find $u_n =$ general n^{th} term from S_n, we take $\boxed{u_n = S_n - S_{n-1}}$

Example:

The sum of the first n terms of a series is given by $S_n = n^2 + 3n$. Show that the terms of the series are in an arithmetic progression.

Solution:

To show that the series is an AP, we need to show $u_n - u_{n-1} =$ common difference $=$ constant

Since we are not given the expression of u_n, we can find it by applying:

$$u_n = S_n - S_{n-1}$$
$$u_n = n^2 + 3n - \left[(n-1)^2 + 3(n-1)\right]$$
$$u_n = 2n + 2$$

Now, consider $u_n - u_{n-1} = [2n+2] - [2(n-1)+2] = 2 = \text{constant}$

Therefore, the series is an AP.

Consecutive terms

AP	GP
If x, y, z are 3 **consecutive** terms in AP, then: $y - x = z - y$ i.e $2y = x + z$	If x, y, z are 3 **consecutive** terms in GP, then: $\frac{y}{x} = \frac{z}{y}$ i.e $y^2 = xz$

Example:

The 9th, 1st and 3rd term of an AP with non zero common difference form 3 consecutive terms of a GP. Find the common ratio of the GP.

Solution:

For the AP, $T_9 = a + 8d, T_1 = a, T_3 = a + 2d$. Since they are consecutive terms, we know

$$r = \frac{T_1}{T_9} = \frac{T_3}{T_1}$$

$$a^2 = (a + 8d)(a + 2d)$$
$$a^2 = a^2 + 10ad + 16d^2$$
$$10ad + 16d^2 = 0$$
$$\Rightarrow d(5a + 8d) = 0$$

Since $d \neq 0, \therefore a = -\frac{8}{5}d$. Hence $r = \frac{a}{a + 8d} = \frac{-\frac{8}{5}d}{-\frac{8}{5}d + 8d} = -\frac{1}{4}$.

Example:

Given that $\frac{1}{y - x}, \frac{1}{2y}, \frac{1}{y - z}$ are consecutive terms of an AP, prove that x, y, z are consecutive terms of a GP.

Solution:

Since $\frac{1}{y - x}, \frac{1}{2y}, \frac{1}{y - z}$ are in AP

$$\therefore 2\left(\frac{1}{2y}\right) = \frac{1}{y - x} + \frac{1}{y - z} = \frac{2y - x - z}{(y - x)(y - z)}$$

$\Rightarrow (y-x)(y-z) = 2y^2 - xy - yz$

$\Rightarrow y^2 + xz - xy - yz = 2y^2 - xy - yz$

$\Rightarrow y^2 = xz$

$\Rightarrow \dfrac{y}{x} = \dfrac{z}{y}$

Therefore, x, y, z are consecutive terms of a GP.

Application based AP/GP questions

Example:

A student puts $10 on 1 january 2009 into a bank account which pays compound interest at a rate of 2% per month on the last day of each month. She puts a further $10 into the account on the first day of each subsequent month.

a) How much compound interest has her original $10 earned at the end of 2 years?
b) How much in total is in the account at the end of 2 years?
c) After how many complete months will the total in the account first exceed $2000?

[2008 A levels]

Solution:
a) We see that

Time:	Jan	Feb 1st	Mar 1st	…
Amt at start:	10	$(1.02\times10)+10$	$(1.02)^2 10+(1.02)10+10$	…
Amt at end:	1.02×10	$1.02(1.02\times10+10)$ $=(1.02)^2 10+(1.02)10$	$1.02\left((1.02)^2 10+(1.02)10+10\right)$ $=(1.02)^3 10+(1.02)^2 10+(1.02)10$	

At the end of 2 years, there would have been 24 months, the total amount of money from the original \$10 = $(1.02)^{24} 10 = 16.084$.

Hence, compound interest from original \$10 is $\$16.084 - \$10 = \$6.084 = \$6.08$ (to 3 sig fig)

b) The total amount of money in the account at the end of the month follows the sum of a GP with first term 1.02×10 and common ratio 1.02, hence total amount of money in account at the end of 2 years.

$$= S_{24} = \frac{1.02 \times 10\left((1.02)^{24} - 1\right)}{(1.02) - 1} = 310.302.$$

c) We want $S_n > 2000 \Rightarrow \dfrac{1.02 \times 10\left((1.02)^{n} - 1\right)}{(1.02) - 1} > 2000$

$$\Rightarrow 510\left((1.02)^n - 1\right) > 2000$$

$$\Rightarrow (1.02)^n > \frac{200}{51} + 1$$

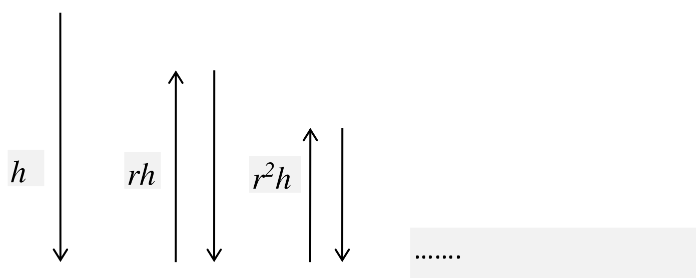
$$\Rightarrow n > \frac{\lg\left(\frac{200}{51} + 1\right)}{\lg(1.02)} = 80.476$$

Thus, least value of n is 81. The account will first exceed \$2000 after 81 months.

Example:

A ball is dropped from a height of h metres. Each time it hits the ground after falling a distance of x metres, it rebounds to a distance of rh metres where r is a positive fraction less than 1. Find the total distance travelled by the ball.

Solution:



Total distance = $h + 2rh + 2r^2h + \cdots = h + 2hr(1 + r + r^2 + \cdots)$

$$= h + 2hr\left(\frac{1}{1-r}\right) = \left(\frac{1+r}{1-r}\right)h$$

Pattern based AP/GP questions

Example:
The positive integers, starting at 1, are grouped into sets containing 1, 2, 4, 8, … integers, as indicated below, so that the number of integers in each set after the first is twice the number of integers in the previous set.

$$\{1\},\ \{2, 3\},\ \{4, 5, 6, 7\}\ \{8, 9, 10, 11, 12, 13, 14, 15\}, \ldots..$$

Write down expressions, in terms of *n*, for
i) the number of integers in the *n*th set,
ii) the first integer in the *n*th set,
iii) the last integer in the *n*th set.

Solution:
We try to write out the number pattern based on what is required.

i) Number pattern for the <u>number of integers</u> in each set goes like this:

1, 2, 4, 8, … which is a G.P.

Therefore, we can apply formula for G.P.: $u_n = ar^{n-1} \Rightarrow$ number of integers in *n*th set $= 1(2)^{n-1}$.

ii) Number pattern for the <u>first integer</u> in each set goes like this:

1, 2, 4, 8, … which is a G.P.

Therefore, we can apply formula for G.P.: $u_n = ar^{n-1} \Rightarrow$ first integer in *n*th set $= 1(2)^{n-1}$.

iii) First integer in $(n+1)$th set $= 1(2)^{(n+1)-1} = 2^n$.

Therefore, last integer in *n*th set
$=$ first integer in $(n+1)$th set -1
$= 2^n - 1$

Common mistakes & Learning points

- $\boxed{u_n = S_n - S_{n-1}}$ can be applied to any series, regardless of it being any kind of progression.
- $\dfrac{S_n}{S_{n-1}}$ does not represent anything. Do not confuse with $\dfrac{u_n}{u_{n-1}}$ which gives us the common ratio if the terms follow a GP.
- $u_1 = S_1$

- **Sequences Vs Series**

Sequence	Series
A sequence of numbers: $\frac{3}{4},\frac{4}{6},\frac{5}{8},\frac{6}{10},\ldots$ Formula defining each term in the above sequence: $u_r=\frac{r+2}{2(r+1)},r\in\mathbb{Z}^+$ **Note:** $\frac{3}{4},\frac{4}{6},\frac{5}{8},\frac{6}{10},\ldots,\frac{n+2}{2(n+1)}$ is a finite sequence. $\frac{3}{4},\frac{4}{6},\frac{5}{8},\frac{6}{10},\ldots$ is an infinite sequence.	A series is the **addition** of the terms in a sequence: i.e. $\frac{3}{4}+\frac{4}{6}+\frac{5}{8}+\frac{6}{10}+\ldots+\frac{n+2}{2(n+1)}$ $=u_1+u_2+\ldots+u_n$ $=\sum_{r=1}^{n}u_r=\sum_{r=1}^{n}\frac{r+2}{2(r+1)}$. **Note:** u_r is the general term for the sequence r is the variable in the summation n is the largest term
2 types of sequences A. <u>"Simple" sequence</u> e.g. $u_r=3r-1$ First term = $u_1=3(1)-1=2$ Second term = $u_2=3(2)-1=5$, etc… B. <u>Recurrence Relation</u> e.g. $u_{r+2}=6u_{r+1}-8u_r$ for $r\in\mathbb{Z}^+$, given $u_1=6$, $u_2=20$. First term = $u_1=6$ Second term = $u_2=20$ Third Term = $u_3=u_{1+2}=6u_{1+1}-8u_1=6u_2-8u_1$ $=6(20)-8(6)=72$, etc… provided if required	A. **Number of terms** Total number of terms in $\sum_{r=m}^{n}u_r$ is $\boxed{n-m+1}$ B. **Sum of constant** $\sum_{r=m}^{n}a=a(n-m+1)$, where a is a constant. C. **Difference of Sums** $\sum_{r=m}^{n}u_r=\sum_{r=1}^{n}u_r-\sum_{r=1}^{m-1}u_r$ e.g. $\sum_{r=35}^{79}u_r=\sum_{r=1}^{79}u_r-\sum_{r=1}^{34}u_r$ D. **Distributive Property of Sums** If a, b and c are constants $\sum_{r=m}^{n}(au_r\pm b)=a\sum_{r=m}^{n}u_r\pm b\sum_{r=m}^{n}1$ E. **Useful formulae** $\sum_{r=1}^{n}r=1+2+\ldots+n=\frac{n(n+1)}{2}$ $\sum_{r=1}^{n}r^2=1^2+2^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$ $\sum_{r=1}^{n}r^3=1^3+2^3+\ldots+n^3=\frac{n^2(n+1)^2}{4}$

Example:

By using the formula $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$, express $\sum_{r=1}^{n}(2r-1)^2$ in terms of n .

Solution:

$$\sum_{r=1}^{n}(2r-1)^2 = \sum_{r=1}^{n}(4r^2-4r+1)$$

$$= 4\sum_{r=1}^{n} r^2 - 4\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$$

$$= 4\cdot\frac{n}{6}(n+1)(2n+1) - 4\cdot\frac{n(n+1)}{2} + n$$

$$= \frac{2n}{3}(n+1)(2n+1) - 2n(n+1) + n$$

$$= \frac{2n}{3}(n+1)(2n+1) - n(2n+1)$$

$$= \frac{n}{3}(2n+1)\left[2(n+1)-3\right] = \frac{n}{3}(2n+1)(2n-1)$$

$$= \frac{n}{3}\left(4n^2-1\right)$$

Example:

The n^{th} term of a series is $2^{n-2}+7n-5$, find the sum of the first N terms.

Solution:

$$\sum_{n=1}^{N}\left(2^{n-2}+7n-5\right) = \sum_{n=1}^{N} 2^{n-2} + 7\sum_{n=1}^{N} n - \sum_{n=1}^{N} 5$$

$$= \left(2^{-1}+2^{0}+\ldots+2^{N-2}\right) + 7(1+2+\ldots+N) - (5+5+\ldots+5)$$

$$= \frac{\frac{1}{2}\left(2^N-1\right)}{2-1} + \frac{7N}{2}(1+N) = \frac{1}{2}\left(2^N-1\right) + \frac{7N}{2}(1+N) - 5N = \frac{1}{2}\left(2^N-1\right) + \frac{7N^2}{2} - \frac{3N}{2}$$

Sequence	Series
Limit of a Sequence For an infinite sequence, the sequence is convergent if: $$\lim_{r\to\infty} u_r = \text{constant} = l$$ i.e. the sequence is convergent if as $r \to \infty, u_r \to l$.	**Limit of a Series** For an infinite series, the series is convergent if: $$\sum_{r=1}^{\infty} u_r = \lim_{n\to\infty}\left(\sum_{r=1}^{n} u_r\right) = \text{constant}$$

Example:

Determine if the following are convergent or divergent.

i) 1, 2, 3, 4, …

ii) $1, -1, 1, -1, \ldots$

iii) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

Solution:

i) This sequence is an AP with first term 1 and common difference 1. Observe that the terms tends to infinity, hence the sequence diverges.

ii) This sequence is a GP with first term 1 and common ratio -1. Observe that the terms do not go to a constant value, hence the sequence diverges.

iii) Observe that the terms tends to zero, hence the sequence converges.

Common mistakes & Learning points

- Useful results

1. $\lim_{n\to\infty}\frac{\text{constant}}{n}=0,\ \lim_{n\to\infty}\frac{\text{constant}}{n!}=0$,
2. $\lim_{n\to\infty}a^n=0,\ \text{if } 0<a<1$,
3. $\lim_{n\to\infty}\frac{1}{a^n}=0,\ \text{if } a>1$.

- $\sum_{r=m}^{n}(U_r)(T_r)\neq\left[\sum_{r=m}^{n}(U_r)\right]\left[\sum_{r=m}^{n}(T_r)\right]$

For eg, $\sum_{r=7}^{79}r^3\neq\left[\sum_{r=7}^{79}r\right]\left[\sum_{r=7}^{79}r^2\right]$

- Generalising numbers

1. Even numbers: $2k, k\in\mathbb{Z}$
2. Odd numbers: $2k-1 \text{ or } 2k+1, k\in\mathbb{Z}$
3. Multiples of n: $kn, k\in\mathbb{Z}$. For eg, all multiples of 3 follow the same general term of $3k$.

Method of Differences

Method of differences occurs when similar terms in a series cancel each other. For eg,

$$\begin{aligned}\sum_{r=1}^{n}[(r-1)^3-r^3] &= [(0)^3-1^3] \\ &\quad +(1)^3-2^3 \\ &\quad +(2)^3-3^3 \\ &\quad +\ldots \\ &\quad +(n-2)^3-(n-1)^3 \\ &\quad +(n-1)^3-n^3] \\ &= -n^3\end{aligned}$$

For method of differences to work:

1 – there must be a **difference** of two (or more) similar terms, e.g. $(r+1)^2-r^2$, $\frac{1}{r+1}-\frac{1}{r-1}$.

2 – there must be a sigma notation, i.e. $\sum$.

There a few scenarios to create the method of differences.

Example: Partial fractions

Express $\frac{r}{(2r-1)(2r+1)(2r+3)}$ in partial fractions. Hence, or otherwise, show that

$\sum_{r=1}^{n}\frac{r}{(2r-1)(2r+1)(2r+3)}=\frac{n(n+1)}{2(2n+1)(2n+3)}$, and find the infinite series $\frac{1}{1\cdot3\cdot5}+\frac{2}{3\cdot5\cdot7}+\cdots$.

Solution:

Let $\frac{r}{(2r-1)(2r+1)(2r+3)}=\frac{A}{2r-1}+\frac{B}{2r+1}+\frac{C}{2r+3}$

By cover up rule,

$$A=\frac{\left(\frac{1}{2}\right)}{\left(2\left(\frac{1}{2}\right)+1\right)\left(2\left(\frac{1}{2}\right)+3\right)}=\frac{1}{16},\quad B=\frac{\left(-\frac{1}{2}\right)}{\left(2\left(-\frac{1}{2}\right)-1\right)\left(2\left(-\frac{1}{2}\right)+3\right)}=\frac{1}{8},$$

and $C=\frac{\left(-\frac{3}{2}\right)}{\left(2\left(-\frac{3}{2}\right)-1\right)\left(2\left(-\frac{3}{2}\right)+1\right)}=-\frac{3}{16}$

$$\therefore \frac{r}{(2r-1)(2r+1)(2r+3)}=\frac{1}{16(2r-1)}+\frac{1}{8(2r+1)}-\frac{3}{16(2r+3)}$$

$$\sum_{r=1}^{n}\frac{r}{(2r-1)(2r+1)(2r+3)}$$

$$=\sum_{r=1}^{n}\left(\frac{1}{16(2r-1)}+\frac{1}{8(2r+1)}-\frac{3}{16(2r+3)}\right)$$

$$=\frac{1}{16}\sum_{r=1}^{n}\left(\frac{1}{2r-1}+\frac{2}{2r+1}-\frac{3}{2r+3}\right)$$

$$\begin{aligned}
=\frac{1}{16}\Bigg[&\frac{1}{1}+\frac{2}{3}-\frac{3}{5}\\
&+\frac{1}{3}+\frac{2}{5}-\frac{3}{7}\\
&+\frac{1}{5}+\frac{2}{7}-\frac{3}{9}\\
&+\ldots\\
&+\frac{1}{2n-3}+\frac{2}{2n-1}-\frac{3}{2n+1}\Bigg]\\
&+\frac{1}{2n-1}+\frac{2}{2n+1}-\frac{3}{2n+3}\Bigg]
\end{aligned}$$

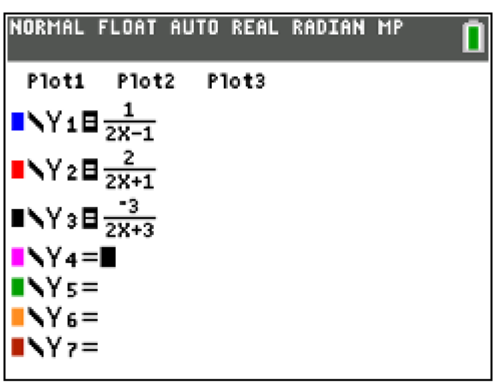
$$=\frac{1}{16}\left(\frac{1}{1}+\frac{2}{3}+\frac{1}{3}-\frac{3}{2n+1}+\frac{2}{2n+1}-\frac{3}{2n+3}\right)$$

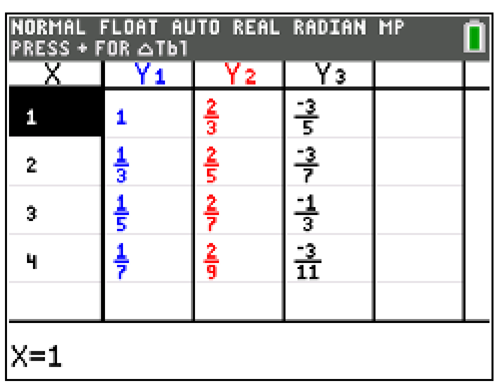
$$=\frac{1}{16}\left(2-\frac{1}{2n+1}-\frac{3}{2n+3}\right)$$

$$=\frac{1}{16}\left(\frac{2(2n+1)(2n+3)-(2n+3)-3(2n+1)}{(2n+1)(2n+3)}\right)$$

$$=\frac{n(n+1)}{2(2n+1)(2n+3)}$$

Do consider using the GC table function when listing out the terms for MOD as it shows us all the terms without us having to substitute values in one by one.





$$\frac{1}{1\cdot3\cdot5}+\frac{2}{3\cdot5\cdot7}+\frac{3}{5\cdot7\cdot9}+\ldots$$
$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r}{(2r-1)(2r+1)(2r+3)}$$
$$=\lim_{n\to\infty}\frac{n(n+1)}{2(2n+1)(2n+3)}$$
$$=\lim_{n\to\infty}\frac{n^2+n}{8n^2+16n+6}$$
$$=\lim_{n\to\infty}\frac{1+\frac{1}{n}}{8+\frac{16}{n}+\frac{6}{n^2}}=\frac{1}{8}$$

Upon getting $\frac{\infty}{\infty}$: divide the expression throughout by n^2 (highest power in this expression) because it is easier to take limits as $n\to\infty$ when n is in the denominator.

Example: Trigonometry

Given $0<\theta<\frac{\pi}{2}$, prove that $\frac{\sin 2(k+1)\theta-\sin 2k\theta}{\sin\theta}=2\cos(2k+1)\theta$, for all $k\in\mathbb{Z}$.

Find $\sum_{r=0}^{N}\cos(2r+1)\theta$, in terms of N and θ. [2010 TJC J2 CT]

Solution:

$$LHS=\frac{\sin 2(k+1)\theta-\sin 2k\theta}{\sin\theta}$$

$$=\frac{2\cos\left[\frac{2(k+1)\theta+2k\theta}{2}\right]\sin\left[\frac{2(k+1)\theta-2k\theta}{2}\right]}{\sin\theta}$$ (by factor formula)

$$=\frac{2\cos(2k+1)\theta\,\sin\theta}{\sin\theta}=2\cos(2k+1)\theta=RHS$$

$$\sum_{r=0}^{N}\cos(2r+1)\theta=\frac{1}{2\sin\theta}\sum_{r=0}^{N}\left[\sin 2(r+1)\theta-\sin 2r\theta\right]$$

$$
\begin{aligned}
=\frac{1}{2\sin\theta}[&\sin 2\theta-\sin 0\\
&+\sin 4\theta-\sin 2\theta\\
&+\sin 6\theta-\sin 4\theta\\
&+\cdots\\
&+\sin 2(N+1)\theta-\sin 2N\theta]
\end{aligned}
$$

$$=\frac{\sin 2(N+1)\theta-\sin 0}{2\sin\theta}=\frac{\sin 2(N+1)\theta}{2\sin\theta}$$

Example: Using ln

Simplify $S_n=\sum_{r=3}^{N-2}\left[7+\ln\left(\frac{r+2}{r}\right)\right]$.

Solution:

$$S_n=\sum_{r=3}^{N-2}\left[7+\ln\left(\frac{r+2}{r}\right)\right]=\sum_{r=3}^{N-2}7+\sum_{r=3}^{N-2}\left[\ln(r+2)-\ln r\right].$$

Now $\sum_{r=3}^{N-2}7=7(N-2-3+1)=7(N-4).$

$$S_n=7(N-4)+\sum_{r=3}^{N-2}\left[\ln(r+2)-\ln r\right]$$

$$
\begin{aligned}
=7(N-4)+[&\ln 5-\ln 3\\
&+\ln 6-\ln 4\\
&+\ln 7-\ln 5\\
&+\ln 8-\ln 6\\
&+\ldots\\
&+\ln(N-1)-\ln(N-3)\\
&+\ln(N)-\ln(N-2)]
\end{aligned}
$$

Note that the cancellation skips one step in this example.

$$=7(N-4)+\ln(N-1)+\ln(N)-\ln 3-\ln 4=7(N-4)+\ln\left(\frac{N(N-1)}{12}\right)$$

Example: Surds

Express $\dfrac{1}{\sqrt{r}+\sqrt{r+2}}$ in the form $a\sqrt{r}+b\sqrt{r+2}$ where a and b are constants.

Hence, show $\dfrac{1}{\sqrt{1}+\sqrt{3}}+\dfrac{1}{\sqrt{3}+\sqrt{5}}+\dfrac{1}{\sqrt{5}+\sqrt{7}}+\ldots+\dfrac{1}{\sqrt{2n+1}+\sqrt{2n+3}}=\dfrac{1}{2}\left[\sqrt{2n+3}-1\right]$

Solution:

Now, $\dfrac{1}{\sqrt{r}+\sqrt{r+2}}=\dfrac{\sqrt{r}-\sqrt{r+2}}{r-(r+2)}=\dfrac{1}{2}\sqrt{r+2}-\dfrac{1}{2}\sqrt{r}$

Hence, $\dfrac{1}{\sqrt{1}+\sqrt{3}}+\dfrac{1}{\sqrt{3}+\sqrt{5}}+\dfrac{1}{\sqrt{5}+\sqrt{7}}+\ldots+\dfrac{1}{\sqrt{2n+1}+\sqrt{2n+3}}$

$$=\sum_{r=0}^{n}\frac{1}{\sqrt{2r+1}+\sqrt{2r+3}}=\frac{1}{2}\sum_{r=0}^{n}\left[\sqrt{2r+3}-\sqrt{2r+1}\right]$$

$$\begin{aligned}
=\frac{1}{2}\Big[&\sqrt{3}-\sqrt{1}\\
&+\sqrt{5}-\sqrt{3}\\
&+\sqrt{7}-\sqrt{5}\\
&+\ldots\\
&+\sqrt{2n+3}-\sqrt{2n+1}\Big]=\frac{1}{2}\left(\sqrt{2n+3}-1\right).
\end{aligned}$$

Example: Substitution and balancing limits

Given that $\sum_{k=1}^{n} k!(k^2+1)=(n+1)!n$, find $\sum_{k=1}^{n-1}(k+1)!(k^2+2k+2)$. [3][2017 DHS Prelims]

Solution:

To use the given result, we need to try and create the general term in the given result.

Hence we replace k by $(k-1)$:

$$\begin{aligned}\sum_{k=1}^{n-1}(k+1)!(k^2+2k+2) &= \sum_{k-1=1}^{k-1=n-1}(k-1+1)!((k-1)^2+2(k-1)+2)\\ &= \sum_{k=2}^{n} k!(k^2+1)\\ &= \sum_{k=1}^{n} k!(k^2+1)-1!(1^2+1)\\ &= (n+1)!n-2\end{aligned}$$

Common mistakes & Learning points

- In general, we

  1. List out the terms to see if there is a simple observable pattern (eg AP, GP etc)
  2. Try and split the terms into separate smaller sums
  3. MOD

- When handling summation, we need to differentiate between solving for it versus using its result.

r is the variable here which helps us identify the sum.

Example: To solve for $\sum_{r=79}^{2n} r$, we list it out according to the limits of *r*, hence we know

$\sum_{r=79}^{2n} r = 79+80+81+...+2n$ which is an AP with first term 79, common difference 1 and having

$2n-79+1=2n-78$ terms. Hence $\sum_{r=79}^{2n} r = \frac{2n-78}{2}[79+2n]=(n-39)(79+2n)$.

n (ending value) is the value which might vary when using the result of the sum.

(not *r*)

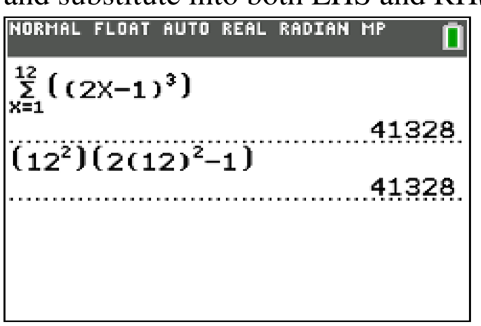
Example: To solve for $\sum_{r=3}^{2n} r^2$ using $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$.

When using the result of the series, the ending limit *n* can be adjusted accordingly.

$$\sum_{r=3}^{2n} r^2 = \sum_{r=1}^{2n} r^2 - 1^2 - 2^2 = \frac{2n(2n+1)(4n+1)}{6} - 5 \quad \text{(replace } n \text{ by } 2n\text{)}$$

- When expressing summation in terms of unknown(s) (eg *n*), we can check our results using GC by substituting values for these unknown(s).

For eg, after solving, we find that $\sum_{r=1}^{n}(2r-1)^3 = n^2(2n^2-1)$. Choose random values of *n* (eg 12) and substitute into both LHS and RHS to check that they are the same.



Binomial Series

1. The binomial series is an infinite series which works as an approximation which is valid for a range of values of x.

$\boxed{(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + \frac{n(n-1)\cdots(n-r+1)}{r!}x^r + \cdots}$ where n is not a positive integer. Range of validity is $|x|<1$.

It is a must to convert the expression in the form $(1+\text{something})^n$ before doing the expansion. This is done by taking out the constant or x term.

E.g. $(2+x)^{-2} = 2^{-2}\left(1+\frac{x}{2}\right)^{-2}$.

In addition, this will allow us to find the range of validity, $\left|\frac{x}{2}\right|<1 \Rightarrow |x|<2$

If we want a series in ascending powers of x, then we ensure that 'x' is in the numerator as in above.

If we want the series to be in descending powers of x, then we take out 'x' so that it is in the denominator, i.e. $(2+x)^{-2} = x^{-2}\left(1+\frac{2}{x}\right)^{-2}$, and the range of validity is $\left|\frac{2}{x}\right|<1 \Rightarrow |x|>2$.

In general, for **ascending** powers of x.

$$\boxed{\begin{gathered}(a+bx)^n = a^n\left(1+\frac{bx}{a}\right)^n = a^n\left[1+n\left(\frac{bx}{a}\right)+\frac{n(n-1)}{2!}\left(\frac{bx}{a}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{bx}{a}\right)^3+\ldots\right]\\ \text{Validity Range: } \left|\frac{bx}{a}\right|<1 \quad ie. \quad |x|<\left|\frac{a}{b}\right|\end{gathered}}$$

For **descending** powers of x.

$$\boxed{\begin{gathered}(a+bx)^n = (bx)^n\left(1+\frac{a}{bx}\right)^n = x^n\left[1+n\left(\frac{a}{bx}\right)+\frac{n(n-1)}{2!}\left(\frac{a}{bx}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{a}{bx}\right)^3+\ldots\right]\\ \text{Validity Range: } \left|\frac{a}{bx}\right|<1 \quad ie. \quad |x|>\left|\frac{a}{b}\right|\end{gathered}}$$

2. **Range of validity** refers to the values of x for which the binomial expansion holds true.

Example: $\sqrt{1+x}=(1+x)^{\frac{1}{2}}\approx 1+\frac{x}{2}-\frac{x^2}{8}$, valid for $|x|<1$

When $x=\frac{1}{2}$: $\text{LHS}=\sqrt{1+\frac{1}{2}}\approx 1.22$, $\text{RHS}=1+\frac{1}{2}\left(\frac{1}{2}\right)-\frac{1}{8}\left(\frac{1}{2}\right)^2\approx 1.21$. Both values are close.

When $x=10$: $\text{LHS}=\sqrt{1+10}\approx 3.32$, $\text{RHS}=1+\frac{1}{2}(10)-\frac{1}{8}(10)^2\approx -6.5$. Both values are very different.

The equation is only true when values of x lie in the range of validity.

Example:

Expand $\left(\frac{1}{x^2}+4\right)^{-2}$ as a series of ascending powers of x, up to and including the term in x^{10}.
State the values of x for which this expansion is valid.
By substituting a suitable value of x, estimate the value of $\frac{1}{104^2}$ to 8 decimal places.

Solution:

$$\left(\frac{1}{x^2}+4\right)^{-2}=\left[\frac{1}{x^2}\left(1+4x^2\right)\right]^{-2}$$

Note that $\frac{1}{x^2}$ is factorised out and not 4 to obtain ascending series.

$$=x^4\left(1+4x^2\right)^{-2}$$

$$=x^4\left[1+(-2)\left(4x^2\right)+\frac{(-2)(-3)}{2!}\left(4x^2\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(4x^2\right)^3+\ldots\right]$$

$$=x^4\left[1-8x^2+48x^4-256x^6+\ldots\right]$$

$$\approx x^4-8x^6+48x^8-256x^{10}$$

Range of validity is $\left|4x^2\right|<1$, i.e. $\left|x^2\right|<\frac{1}{4}\Rightarrow|x|<\frac{1}{2}$.

Comparing $\frac{1}{104^2}$ with $\left(\frac{1}{x^2}+4\right)^{-2}$:

We can rewrite $\frac{1}{104^2}=(4+100)^{-2}=\left(4+\frac{1}{0.1^2}\right)^{-2}$, hence we substitute $x=0.1$.

Check!
It is important to do a quick check on the value of x that we are substituting with the range of validity to make sure it falls within!

i.e $\dfrac{1}{104^2} \approx (0.1)^4 - 8(0.1)^6 + 48(0.1)^8 - 256(0.1)^{10}$
= 0.00009245 (to 8 decimal places)

Example:
Expand $\dfrac{1}{(1-2x)^4}$ as a series of ascending powers of x, where $|x| < \dfrac{1}{2}$, up to and including the term in x^3. Find, in its simplest form, the coefficient of x^r.

Solution:

$$\frac{1}{(1-2x)^4} = (1-2x)^{-4}$$

$$= 1 + (-4)(-2x) + \frac{(-4)(-5)}{2!}(-2x)^2 + \frac{(-4)(-5)(-6)}{3!}(-2x)^3 + \ldots$$

$$= 1 + 8x + 40x^2 + 160x^3 + \ldots$$

Coefficient of x^r

$$= \frac{(-4)(-4-1)(-4-2)\ldots[-4-(r-1)]}{r!}(-2)^r$$

$$= \frac{(-4)(-5)(-6)\ldots(-r-3)}{r!}(-1)^r(2)^r$$

$$= \frac{(-1)^r[4.5.6\ldots(r+3)](-1)^r 2^r}{r!}$$

$$= \frac{(-1)^{2r}[1.2.3.4.5\ldots\ldots r.(r+1)(r+2)(r+3)]}{r!(1.2.3)}2^r$$

$$= \frac{(r+1)(r+2)(r+3)}{6}2^r$$

Example: [2017 ACJC Prelims]

i) Expand $\left(k+x\right)^n$, in ascending powers of x, up to and including the term in x^2, where k is a non-zero real constant and n is a negative integer. [3]

ii) State the range of values of x for which the expansion is valid. [1]

iii) In the expansion of $\left(k+y+3y^2\right)^{-3}$, the coefficient of y^2 is 2. By using the expansion in (i), find the value of k. [3]

Solution:

i)

$$\begin{aligned}\left(k+x\right)^n &= k^n\left(1+\frac{x}{k}\right)^n \\ &= k^n\left(1+n\left(\frac{x}{k}\right)+\frac{(n)(n-1)}{2!}\left(\frac{x}{k}\right)^2+\ldots\right) \\ &= k^n\left(1+\frac{n}{k}x+\frac{(n)(n-1)}{2k^2}x^2+\ldots\right)\end{aligned}$$

ii) Validity occurs when

$$\left|\frac{x}{k}\right|<1 \Rightarrow |x|<|k|$$

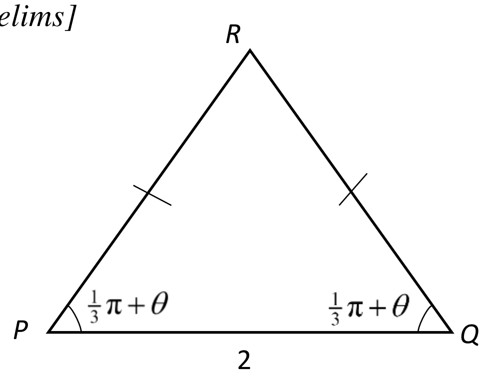
$$\therefore -|k|<x<|k|$$

iii) Let $x=y+3y^2$ and $n=-3$:

$$\begin{aligned}&\left(k+y+3y^2\right)^{-3} \\ &= k^{-3}\left(1+\frac{(-3)}{k}\left(y+3y^2\right)+\frac{(-3)(-4)}{2k^2}\left(y+3y^2\right)^2+\ldots\right) \\ &= k^{-3}\left(1-\frac{3}{k}y-\frac{9}{k}y^2+\frac{6}{k^2}y^2+\ldots\right) \\ &\Rightarrow k^{-3}\left(-\frac{9}{k}+\frac{6}{k^2}\right)=2 \Rightarrow 2k^5+9k-6=0\end{aligned}$$

$\therefore k=0.642$ (to 3 sf)

Example: [2017 DHS Prelims]

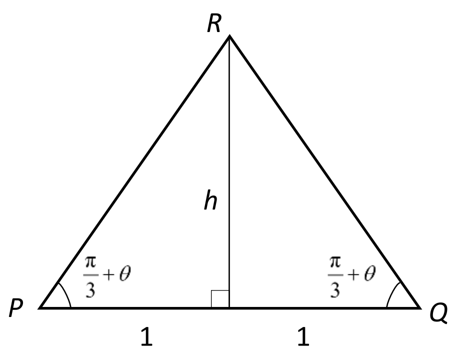


In the isosceles triangle PQR, $PQ = 2$ and the angle QPR = angle PQR $=\left(\frac{1}{3}\pi+\theta\right)$ radians. The area of triangle PQR is denoted by A.

Given that θ is a sufficiently small angle, show that $A=\dfrac{\sqrt{3}+\tan\theta}{1-\sqrt{3}\tan\theta}\approx a+b\theta+c\theta^2$,

for constants a, b and c to be determined in exact form. [5]

Solution:



Observe that $h=\tan\left(\dfrac{\pi}{3}+\theta\right)$.

$$A=\frac{1}{2}(2)\tan\left(\frac{\pi}{3}+\theta\right)=\tan\left(\frac{\pi}{3}+\theta\right)$$

$$=\frac{\tan\left(\frac{\pi}{3}\right)+\tan\theta}{1-\tan\left(\frac{\pi}{3}\right)\tan\theta}=\frac{\sqrt{3}+\tan\theta}{1-\sqrt{3}\tan\theta}\ \text{(shown)}$$

$$\approx\frac{\sqrt{3}+\theta}{1-\theta\sqrt{3}}\quad(\tan\theta\approx\theta\text{ as }\theta\text{ is small})$$

$$=\left(\sqrt{3}+\theta\right)\left(1-\theta\sqrt{3}\right)^{-1}$$

$$\approx\left(\sqrt{3}+\theta\right)\left(1+\theta\sqrt{3}+3\theta^2\right)$$

$$=\sqrt{3}+4\theta+\left(4\sqrt{3}\right)\theta^2$$

Common mistakes & Learning points

- The 'x' term can comprise of more than 1 expression.

For eg, $\left(1+2x-x^2\right)^{\frac{1}{2}}=\left[1+\left(2x-x^2\right)\right]^{\frac{1}{2}}=1+\frac{1}{2}\left(2x-x^2\right)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\left(2x-x^2\right)^2+\ldots$

- When expanding series with a few terms, factorise to create the "1' term.

For eg, $\left(2+x-x^3\right)^{\frac{1}{3}}=2^{\frac{1}{3}}\left[1+\left(\frac{x}{2}-\frac{x^3}{2}\right)\right]^{\frac{1}{3}}=2^{\frac{1}{3}}\left[1+\frac{1}{3}\left(\frac{x}{2}-\frac{x^3}{2}\right)+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}\left(\frac{x}{2}-\frac{x^3}{2}\right)^2\right]+\ldots$

DO NOT DO THIS:

$$\left(2+x-x^3\right)^{\frac{1}{3}}=\left(1+\left(1+x-x^3\right)\right)^{\frac{1}{3}}=\left[1+\frac{1}{3}\left(1+x-x^3\right)+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}\left(1+x-x^3\right)^2\right]+\ldots$$

The above will expand indefinitely.

- Series approximations are generally more accurate when there are more terms listed. The value of x substituted will also affect the accuracy.